Mathability and Mathematical Cognition

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***Abstract*—In this paper, we consider some educational aspects related to mathability. Our main goal is to approach an answer to the question what are computer assisted ways to assimilate mathematical knowledge and possess mathematical abilities.**

***Keywords*—Mathability; Computer Aided Education; Cognitive Abilities.**

INTRODUCTION

In the paper [12], some educational aspects related to the concept of mathability were considered. Among others, it was analysed how the well known taxonomy of educational objectives corresponds to a quantification of the mathability of devices (cf. [8] and [18]). A broad discussion made us formulate a question about the future of mathematical education, namely, how deep one has to know and understand the foundation of mathematics in order to apply modern machines with high mathability level to solve problems related to mathematics (for specific examples and explanation see [2], [3], [4], [9], [15], [16] and [17]). Is it necessary to teach students using classical methods and enforce them to derive, for instance integrals, on their own? Is it correct to let students use mathematical application, computers and other machines to compute partial results in order to solve a complex problem assuming that they do not understand some complicated definitions deeply enough? If so, where is the border between what every engineer should know and what is understandable and useful for scientists rather?

The main goal of the paper is to come closer to an answer to the question what are computer assisted ways to assimilate mathematical knowledge and possess mathematical abilities and how they are related to the ability of applying learnt patterns and procedures, in typical and problematic situations (see also [5], [6], [7] and [19]). Thanks to the answer it will be possible in the future to discuss the minimal knowledge and abilities which each engineer should possess, namely what is the foundation of mathematics enabling people to apply mathematics successfully using variety of machines and mathematical applications. The first part of the paper presents models of computer assisted self-education. We describe some strategies of approaching a solution of a given problem. Moreover, we pay attention to some typical habits of the young generation related to getting information from the Internet. In part two, we show examples of classical mentor-related teaching mathematics aided with Wolfram Mathematica in order to prove usefulness of high mathability level machines in logical thinking training.

I. MATHEMATICAL SELF-EDUCATION AIDED WITH IT

Four groups of students were investigated from September 2015 to June 2016. The first one consisted of 15 students of the bachelor second year of mathematics with a financial specialization. The second one was formed with 24 students of year one of engineering informatics. Students of both groups study at Kazimierz Wielki University in Bydgoszcz. The members of the two remaining groups were pupils of a lower secondary school in Bydgoszcz (Gimnazjum 50). In general, they were given some problematic tasks to solve having no clues how to do it and usually not knowing even any basic notion in the topic.

I.1. Ways of assimilation

The first considered group was introduced to the theory of mathability. The students tasks were:

1. to formulate a mathematical problem, solve it and present an exemplary solution underlying the methodology of gaining the necessary knowledge leading to the solution;
2. to solve a problem given by an expert.

Five out of fifteen students were deeply acquainted with the learnt material, they understood the basic notions and the related method. They were able to use the method in similar cases. They applied a comparable way of studying. For instance, one of the students problems was: using bisection method, solve the equation $x^3 - x^2 + 2x - 1 = 0$. It should be mentioned that numerical methods of analysis are not known to the students of the financial profile. To solve the equation the student started with analysing solutions of two similar equations in order to understand the method. Then he read theoretical description of the bisection method for solving nonlinear equation. Next, he applied the method to the given equation. The only source of the knowledge was Internet:

– http://www.kosiorowski.edu.pl/wp-content/uploads/2014/09/IS-MetNum-W-S-5-Przyblizone-metody.pdf

– http://fizyczna.chem.pg.edu.pl/documents/175260/14212622/smo_sem_014.pdf
– http://eff10.internetdsl.tpnet.pl/programowanie/mz_fun/pages/bisection.htm

The student did not use any computer application. After the student presented the solution he was asked to solve the equation $3x + \sin x - e^x = 0$ in order to check whether he understood the method and learnt it efficiently. The student proved his deep knowledge. Thanks to the above observation it is easy to sketch a model of self-education aided with IT. Basing on the previous knowledge and abilities one can find information helpful for solving the given problem. The information consists of:

– exemplary solutions showing the appropriate algorithm,
– theoretical background –- further details of the explanation.

The order of the above two points can be inverse. As the next step one can:

– repeat the observed method to work out their own solution in a typical case,
– next, repeat the observed method, in a new or problematic case.

Hence it is possible to build a gradual scheme of learning and self-education aided with IT in order to solve a given problem:

Proposition 1. *Model of constructive self-education*

A. assimilation knowledge - searching for the knowledge and examples,

B. understanding notions, methods, examples,

C. following the way of the analysed solution,

D. finding a solution of the original problem,

E. reflection and assessing the result and method.

One can easily observed that the taxonomy above is close to the Bloom's taxonomy of educational objectives (for more details see [8], [12]), however the order of objectives is different. It is worthwhile to mention that the described taxonomy refers to a cognitive model of constructive education and self-education studied already, among others, by J. Dewey and J. Bruner (cf. [13], [14], [10] and [11]).

It could be compared to the model of a problematic method of teaching (i.e. teaching by formulating the problematic case and discovering ways of solutions, assessing them and choosing the most suitable, adequate and optimal one), one of the most difficult method of teaching and learning. Usually only good students could follow the reasoning presented by other students. Nowadays, the constructive methods of education with IT is applied very often by any student or pupil not only in mathematics. (For some more details on conceptual understanding and procedural knowledge we refer to [20] and [1].)

I.2. Superficiality

Students of the second group were asked to explain with details how two sorted tables could be merged in the MergeSort method. A correct and full answer should describe in details the way of choosing consecutive elements of both tables to copy them to an additional table, underlying priority of the chosen elements and describing the way of shifting indexes of the tables cells which are analysed in each step. It should be marked that the input tables should be sorted, however simple merging could give an unsorted table. The exercise should prepare students to write a computer application sorting a given list of real numbers.

The majority of students used the explanation of Wikipedia [see: https://en.wikipedia.org/wiki/Merge_sort, originally students used a Polish language version: https://pl.wikipedia.org/wiki/Sortowanie_przez_scalanie].

Hence the most frequent answer was: "Merge sort is an efficient, general-purpose, comparison-based sorting algorithm. Most implementations produce a stable sort, which means that the implementation preserves the input order of equal elements in the sorted output. Conceptually, a merge sort works as follows:

1. Divide the unsorted list into n sublists, each containing 1 element (a list of 1 element is considered sorted).
2. Repeatedly merge sublists to produce new sorted sublists until there is only 1 sublist remaining. This will be the sorted list."

They read only a short explanation which is, in fact, an introduction to the detailed description. They did not understand the essence of the procedure. Although later the topic was investigated during regular classes (computer laboratory), having the same question on the exam they limited their answer to the sketchy content of a single website. It proves that in several cases constructive methods of self-education fail. The first students tasks – learning by discovering – had the greatest influence on their knowledge. The further explanation given by the teacher (expert) did not improve their acquaintance significantly. We can easily come to the conclusion the young generation is usually content with sketchy solution with no deeper understanding or reflection.

I.3. Association to prior knowledge system

The third group consisted of 56 pupils aged $13 - 14$ (class one of junior high school). They were divided into 8 tasks groups. The common aim for all groups was to present numbers 47 and 126 using the Fibonacci coding. It is necessary to mention that at school pupils had already learnt binary coding. Right before the tasks they had been presented Fibonacci sequence and its influence to the golden ratio.

The first brief explanation, following Wikipedia, can be: "Fibonacci coding is a universal positional code which encodes positive integers into binary code words. It is one example of representations of integers based on Fibonacci numbers."

Originally pupils used the Polish webpage https://pl.wikipedia.org/wiki/System_Fibonacciego (giving a different explanation than the English one). Further reading gives the next important information:

– no code word can contain two consecutive 1, i.e. the "11" is not allowed,
– the Fibonacci system is defined as: 0, 1, 1, 2, 3, 5, 8, ... and the two first number are never taken into consideration.

Such a method allows to present any positive natural number in a unique form. For instance:

$1000_F = 5$,
$1000101_F = 25$,
$10010010_F = 44$.

Usually it is useful to inverse the order of digits and mark the end of the full number with additional "1". Hence each number in Fibonacci coding ends with double 1 which means the end of its description. It is applicable to write a sequence of numbers. For instance:

$$\begin{aligned} 0001110100011010010011 &= 00011\ 10100011\ 010010011_F \\ &= (5, 25, 44). \end{aligned}$$

This method is corresponding to the one described at: https://en.wikipedia.org/wiki/Fibonacci_coding, where "*each code word ends with* '11' *and contains no other instances of* '11' *before the end.*"

Pupils presented variety of results basing on the Internet descriptions similar to the ones mentioned above.

Solutions of the first sort, at the first glance, seemed to be correct. They presented 47 as 10010110 and 126 as 111100100_F or 111100011_F. It is visible, that pupils read the description not carefully enough. Omitting the rule that "11" is not allowed they did not ensure the uniqueness of the representation. They had found main key words: "Fibonacci" and "binary coding", however they applied it incorrectly.

Another method of solution gave evidently wrong results which was not realised by the pupils. They had misunderstood the description: "*Fibonacci coding is a universal code which encodes positive integers into binary code words.[...] representations of integers based on Fibonacci numbers.*" First, basing on their prior knowledge on binary coding, they coded 47 as 101111_2 and 126 as 1111110_2, next they interpreted the representation using Fibonacci numbers:

$$\begin{aligned} 47 = 101111_2 &= F1 + F2 + F3 + F4 + F6 \\ &= 1 + 2 + 3 + 5 + 13 = 24 \end{aligned}$$

and

$$\begin{aligned} 126 = 1111110_2 &= F2 + F3 + F4 + F5 + F6 + F7 \\ &= 2 + 3 + 5 + 8 + 13 + 21 = 57. \end{aligned}$$

It is easy to observe that students had read the definition carefully but they did not follow any solved examples. It shows how misleading Internet descriptions can be and how far such an introduction can be from any text in any school book which, in general, is methodically well organized.

Summarizing, only two out of eight groups presented correct results of this exercise, i.e. $46 = 10100000_F$ and $126 = 1010000100_F$, although all the pupils were smart and trained in logical thinking. Independently of the way of solution they had chosen they usually made mistakes since they did not pay enough attention to details. We may conclude that appropriate mathematical foundation, the ability of reading with understanding and selecting proper content have the same importance. However, selection of information is the weakest point of the procedure. Hence, the school and academic mathematical education should evolve towards abilities of reading, selecting and concluding (for a more detailed description we refer to the papers [20]). Then, chosen the common foundation of mathematical knowledge, we can believe in success of computer assisted education.

II. Mathematical Mentoring Aided with IT

High mathability level devices enable to exemplify abstract mathematical notions. On the other hand applying IT in mathematical education yields developing creative thinking and even enforce students to well-considered actions, since it is necessary to plan which system or application and in which way should be used in order to obtain a required solution. In this part we focus on such a controlled usage of high mathability level applications.

Between October 2015 and May 2016 two independent investigations were carried out in a group of pupils of High School No 6 in Bydgoszcz and among students of year two of mathematics at Kazimierz Wielki University in Bydgoszcz. Applied mathematical computer applications facilitated understanding basic notions and their usage in further reasoning.

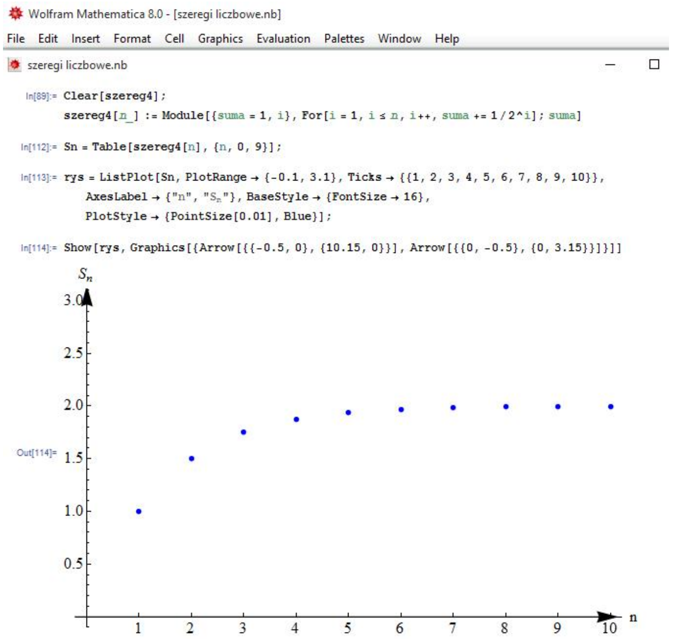
Pupils of High School No 6 from a class with mathematical profile took part in facultative classes throughout one semester. The group consisted of those 20 students of year two who were eager to broaden their mathematical knowledge. The general topic of discussion were chosen problems of mathematical analysis which are introduced on a regular basis as a part of academic material. Computer assisted methods used to gain new knowledge enabled students to obtain surprising results in understanding problems outreaching school material.

II. 1. Computer assisted discovery

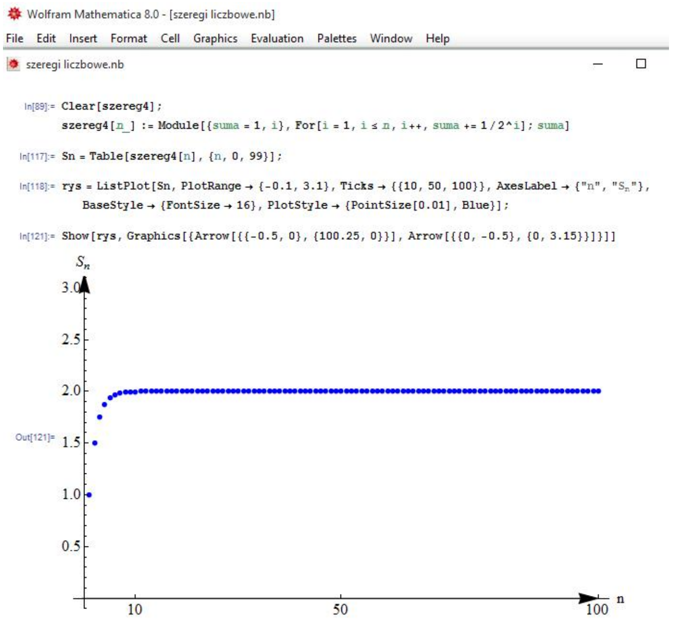
One of problems difficult to understand in mathematical analysis is convergence of a series. After introducing the notion of a limit of a sequence the young people were asked to calculate a sum of elements of a sequence of the form:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = ?.$$

Obviously, there was no problem to compute a sum of any finite partial sum of the given form. It was difficult to discover and understand that such an infinite unique sum exists and can be finite. To exemplify the notions of a series and its convergence, Wolfram Mathematica was applied to compute consequitive partial sums of the above form. Thanks to that pupils could easily interpret partial sums on the graph and understand their meaning. Pictures 1 and 2 present chosen steps of creating partial sums.



Picture 1



Picture 2

Basing on the obtained graphical results, pupils were able to formulate convergence criterion. They realized on their own that the infinite sum would exist if the sequence of partial sums was convergent. Pupils noticed that the infinite sum would be exactly a finite limit of the partial sum sequence. This way, not knowing the notion of a geometric series, students of high school designated the required sum:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2.$$

In order to check if the method was understood correctly, the pupils were asked to find individually a sum of the form:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = ?.$$

Similarly, not knowing the notion of harmonic series, applying graphs in Wolfram Mathematica, pupils unequivocally stated that the required sum did not exist. They correctly interpreted obtained graphs of chosen partial sums.

Undoubtedly, applying computer aided methods helped pupils to interpret partial results, boosted their creative and logical thinking. They discovered and formulated convergence criterion and solved the abstruse problem.

II. 2. Computer assisted interpretation

Another group of the high school students had already learnt the notion of derivative and were able to derive it easily for any elementary function. During facultative classes they were presented the notion of Taylor series and the method of Taylor approximation of a given function.

First, pupils were acquainted with the formula of the approximation of $f : D \to \mathbb{R}$ at a point x_0, where $D \subseteq \mathbb{R}$ is a nonempty set, x_0 is an interior point of D and f is infinitely many times differentiable at x_0:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots$$

and for $x_0 = 0$, i.e. with Macluarin series:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(x_0)}{3!}x^3 + \cdots$$

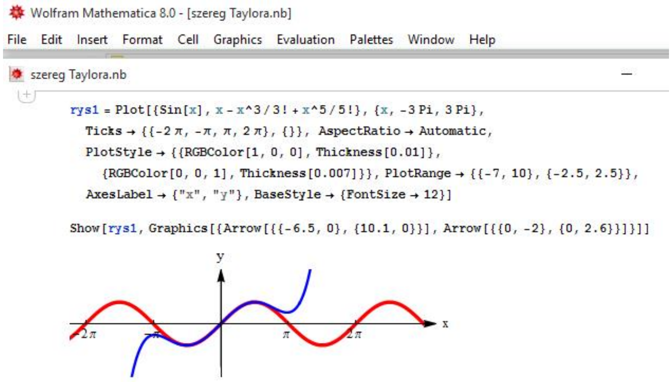
Asked to approximate the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$ with Maclaurin series, the pupils had no problems to derive the correct solution:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots .$$

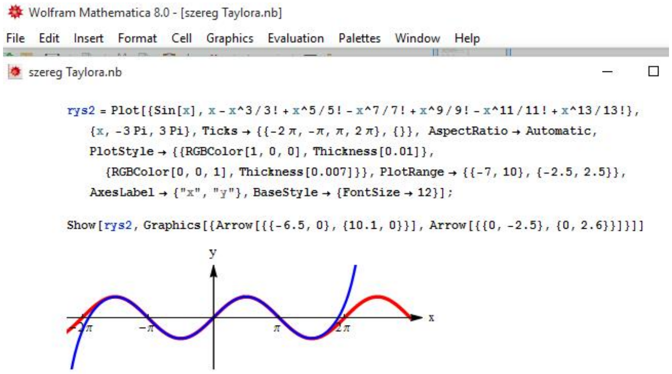
The difficulty of the task was to understand the obtained result and interpret it. Neither were they able to interpret the approximation of a form:

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Again, application of Wolfram Mathematica enabled students to observe the meaning of results thanks to a graphical interpretation. They could easily draw graphs of sinus as well as graphs of polynomial functions of different ranks. Pictures 3 and 4 present graphs of polynomial of ranks 5 and 13, prepared by students.



Picture 3



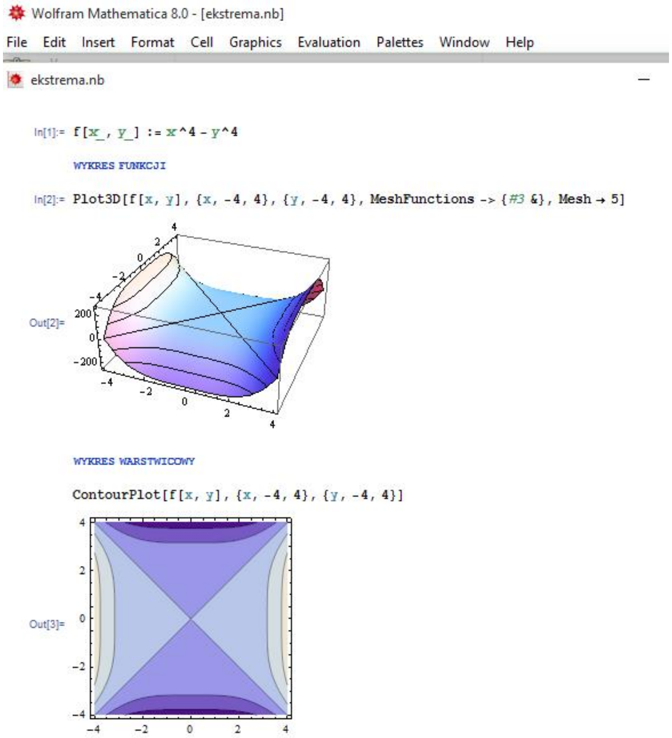
Picture 4

Based on the graphs pupils understood the considered problem and could interpret achieved result. They discovered and formulated the conclusion that approximation of sinus with Maclaurin's series gave a chance to present trigonometric function locally as a simple polynomial function. They observed that the higher is the polynomial rank the better approximation of the function could be obtained. Although the investigated problem was computationally simple it was very abstract with incomprehensible interpretation. The solution was revealed simple and logical with the use of high mathability level application.

II. 3. Computer assisted proofs

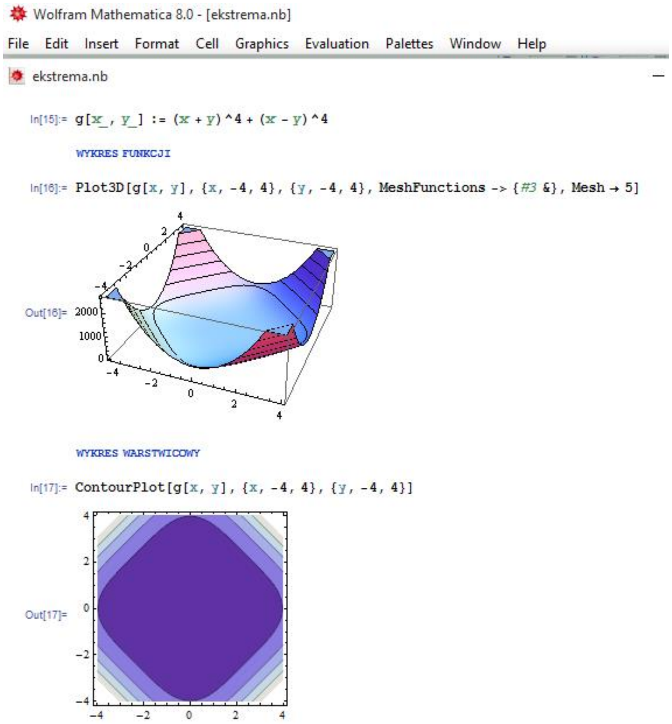
Students of mathematics were investigated during their regular classes of mathematical analysis. They had learnt the definition and the algorithm of determining a local extremum of a function of two variables. According to the algorithm they estimated values of the determinant of partial derivatives of the second order for all points in which the first order partial derivatives were equal to zero. If the obtained determinant value in an examined point was positive students continued investigation and estimated minimum or maximum of the function. If the determinant value was negative they stated that in the given point there was no extremum. The task was to train students to make a decision when the determinant was equal to zero and well known algorithm gave no explicit answer. Applying the definition of the local extrema appeared to be extremely difficult method.

Once again, application of Wolfram Mathematica helped students to solve the problem. First of all they draw graph of the given function. Next, they draw layer graph of the function in the neighbourhood of the investigated point (in which the determinant was equal to zero). Pictures 5 an 6 present graphs drawn by students for functions $f, g : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^4 - y^4$ and $g(x, y) = (x + y)^4 + (x - y)^4$.



Picture 5

Students had computed the point $P(0, 0)$ where the determinant was equal to zero both for the functions f and g. Graph of the functions in Wolfram Mathematica showed students behaviour of the functions in any neighbourhood of point P. For the function $f(x, y) = x^4 - y^4$ students observed that in each such neighbourhood there exists a point for which the value of the function is positive and there exists a point for which the value of the function is negative. Hence, they concluded that here is no extremum in $P(0, 0)$. Analysing the graph of the function $g(x, y) = (x+y)^4 + (x-y)^4$ they stated that for any point in the neighbourhood of P the values of the function g were greater than in P, hence there was a local minimum in P. After such graphical analysis students were able to prove the observed properties theoretically.



Picture 6

Application of mathability tools proved to be useful in conducting students logical thinking when the problem leads behind a standard, typical computation algorithm.

Summarising

Contemporary tools of cognitive infocommunication and facility for getting information throughout instant searching for key words have already changed the way of human cognition and knowledge assimilation. Modern mathematical and technical education should fit the new habits and it is realisable thanks to high mathability level devices and applications. However, at the moment uncontrolled computer assisted self-education can be risky since lack of accuracy and sketchy solutions are characteristic for the young generation. It is necessary to pay more attention to abilities of selection and assessment of gathered information as well as reflection on the obtained result. On the other hand computer assisted mentoring is a valuable method of discovering and applying mathematics, technics and other sciences.

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References

[1] "Mathematical power for all students, k-12. pinellas county schools division of curriculum and instruction." pp. 20–22, 2002.

[2] P. Baranyi, A. Csapo, and P. Varlaki, "An overview of research trends in coginfocom," in *18th International Conference on Intelligent Engineering Systems (INES)*. IEEE, 2014, pp. 181–186.

[3] P. Baranyi and A. Gilányi, "Mathability: emulating and enhancing human mathematical capabilities," in 4^{th} *IEEE International Conference on Cognitive Infocommunications (CogInfoCom)*. IEEE, 2013, pp. 555–558.

[4] P. Baranyi, A. Csapo, and G. Sallai, *Cognitive Infocommunications (CogInfoCom)*. Springer, 2015.

[5] P. Biró and M. Csernoch, "Deep and surface metacognitive processes in non-traditional programming tasks," in 5^{th} *IEEE Conference on Cognitive Infocommunications (CogInfoCom)*. IEEE, 2014, pp. 49–54.

[6] ——, "The mathability of computer problem solving approaches," in 6^{th} *IEEE Conference on Cognitive Infocommunications (CogInfoCom)*. IEEE, 2015.

[7] ——, "The mathability of spreadsheet tools," in 6^{th} *IEEE Conference on Cognitive Infocommunications (CogInfoCom)*. IEEE, 2015.

[8] B. S. Bloom, *Taxonomy of Educational Objectives: The Classification of Educational Goals*. Susan Fauer Company, Inc., 1956.

[9] G. G. Borus and A. Gilányi, "Solving systems of linear functional equations with computer," in 4^{th} *IEEE International Conference on Cognitive Infocommunications (CogInfoCom)*. IEEE, 2013, pp. 559–562.

[10] J. S. Bruner, *The process of education*. Cambridge, Mass: Harvard University Press, 1960.

[11] J. S. Bruner, R. R. Olver, and P. M. Greenfield, *Studies in cognitive growth*. Wiley, 1966.

[12] K. Chmielwska and A. Gilányi, "Mathability and computer aided mathematical education," in 6^{th} *IEEE Conference on Cognitive Infocommunications (CogInfoCom)*. IEEE, 2015, pp. 473–477.

[13] J. Dewey, *How we think*. MA: DC Heath, 1910.

[14] J. Dewey, A. W. Moore, H. C. Brown, G. H. Mead, B. H. Bode, H. W. Stuart, J. H. Tufts, and H. M. Kallen, *Creative intelligence: Essays in the pragmatic attitude*. H. Holt, 1917.

[15] A. Gilányi, N. Merentes, and R. Quintero, "Mathability and an animation related to a convex-like property," in 7^{th} *IEEE Conference on Cognitive Infocommunications (CogInfoCom)*. IEEE, 2016.

[16] A. Gilányi, "Charakterisierung von monomialen Funktionen und Lösung von Funktionalgleichungen mit Computern," Diss., Universität Karlsruhe, Karlsruhe, Germany, 1995.

[17] ——, "Solving linear functional equations with computer," *Math. Pannon.*, vol. 9, no. 1, p. 57–70, 1998.

[18] B. Niemierko, *Między oceną szkolną a dydaktyką*. Bliżej dydaktyki, Warszawa, 1999.

[19] B. Szi and A. Csapo, "An outline of some human factors contributing to mathability research," in 5^{th} *IEEE Conference on Cognitive Infocommunications (CogInfoCom), 2014*. IEEE, 2014, pp. 583–586.

[20] M. Török, M. J. Tóth, and A. Szöllősi, "Foundations and perspectives of mathability in relation to the coginfocom domain," in 4^{th} *IEEE International Conference on Cognitive Infocommunications (CogInfoCom)*. IEEE, 2013, pp. 869–872.